Vector Expansion using Base Vectors

Having defined an orthonormal set of base vectors, we can express **any** vector in terms of these unit vectors:

$$\mathbf{A} = A_x \, \hat{a}_x + A_y \, \hat{a}_y + A_z \, \hat{a}_z$$

Note therefore that any vector can be written as a sum of three vectors!

* Each of these three vectors point in one of the **three orthogonal directions** $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_z$.

* The **magnitude** of each of these three vectors are determined by the scalar values A_x, A_y, and A_z.

* The values A_x, A_y, and A_z are called the **scalar components** of vector **A**.

* The vectors $A_x \, \hat{a}_x$, $A_y \, \hat{a}_y$, $A_z \, \hat{a}_z$ are called the **vector components** of **A**.

Q: *What the heck are scalar the components A_x, A_y, and A_z, and how do we determine them ??*

A: Use the **dot product** to evaluate the expression above !

Begin by taking the **dot product** of the above expression with unit vector $\hat{a}_x$:

$$\begin{aligned}\mathbf{A}\cdot\hat{a}_x &= \left(A_x\,\hat{a}_x + A_y\,\hat{a}_y + A_z\,\hat{a}_z\right)\cdot\hat{a}_x \\ &= A_x\,\hat{a}_x\cdot\hat{a}_x + A_y\,\hat{a}_y\cdot\hat{a}_x + A_z\,\hat{a}_z\cdot\hat{a}_x\end{aligned}$$

But, since the unit vectors are **orthogonal**, we know that:

$$\hat{a}_x\cdot\hat{a}_x = 1 \qquad \hat{a}_y\cdot\hat{a}_x = 0 \qquad \hat{a}_z\cdot\hat{a}_x = 0$$

Thus, the expression above becomes:

$$A_x = \mathbf{A}\cdot\hat{a}_x$$

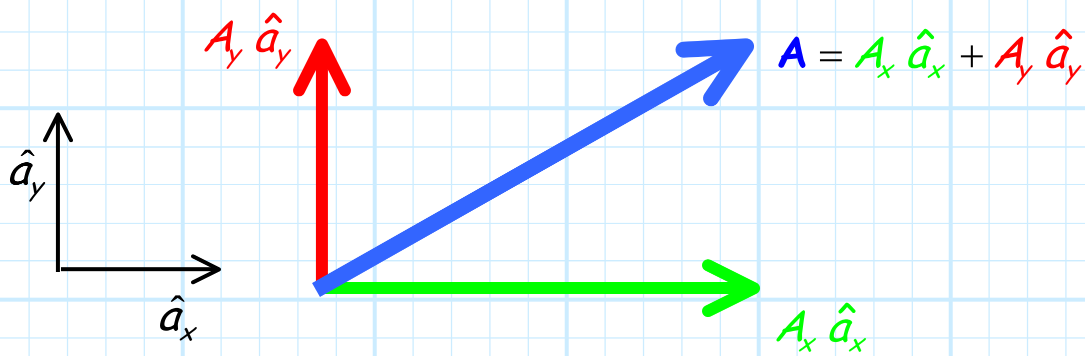
In other words, the scalar component A_x is just the value of the **dot product** of vector **A** and base vector $\hat{a}_x$. Similarly, we find that:

$$A_y = \mathbf{A}\cdot\hat{a}_y \quad \text{and} \quad A_z = \mathbf{A}\cdot\hat{a}_z$$

Thus, **any vector** can be expressed specifically as:

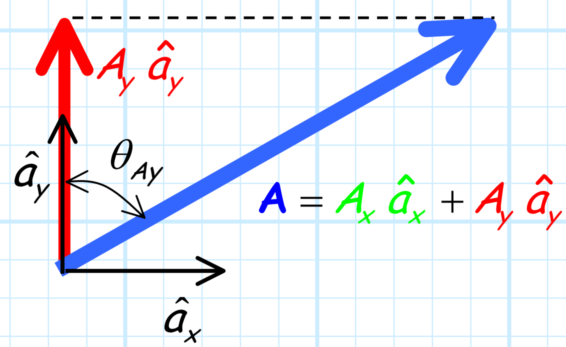
$$\begin{aligned}\mathbf{A} &= \left(\mathbf{A}\cdot\hat{a}_x\right)\hat{a}_x + \left(\mathbf{A}\cdot\hat{a}_y\right)\hat{a}_y + \left(\mathbf{A}\cdot\hat{a}_z\right)\hat{a}_z \\ &= A_x\,\hat{a}_x + A_y\,\hat{a}_y + A_z\,\hat{a}_z\end{aligned}$$

We can demonstrate this vector expression geometrically.

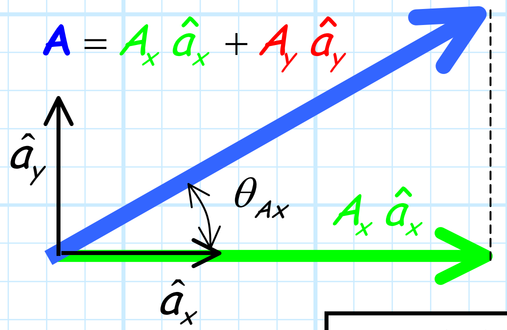


Note the length (i.e., magnitude) of vector **A** can be related to the length of vector $A_y\,\hat{a}_y$ using trigonometry:

$$A_y = |\mathbf{A}|\cos\theta_{Ay}$$



Likewise, we find that the scalar component A_x is related to $|\mathbf{A}|$ as:



$$A_x = |\mathbf{A}| \cos \theta_{Ax}$$

From this geometric interpretation, we can see why we often refer to the scalar component A_x as the **scalar projection** of vector **A onto** vector (direction) $\hat{a}_x$.

Likewise, we often refer to the vector component $A_x\, \hat{a}_x$ as the **vector projection** of vector **A onto** vector (direction) $\hat{a}_x$.

*As **you** may have already noticed, the **scalar component** A_x, which we determined geometrically, can likewise be expressed in terms of a **dot product**!*

$$\begin{aligned} A_x &= |\mathbf{A}| \cos \theta_{Ax} \\ &= |\mathbf{A}|\, |\hat{a}_x| \cos \theta_{Ax} \\ &= \mathbf{A} \cdot \hat{a}_x \end{aligned}$$

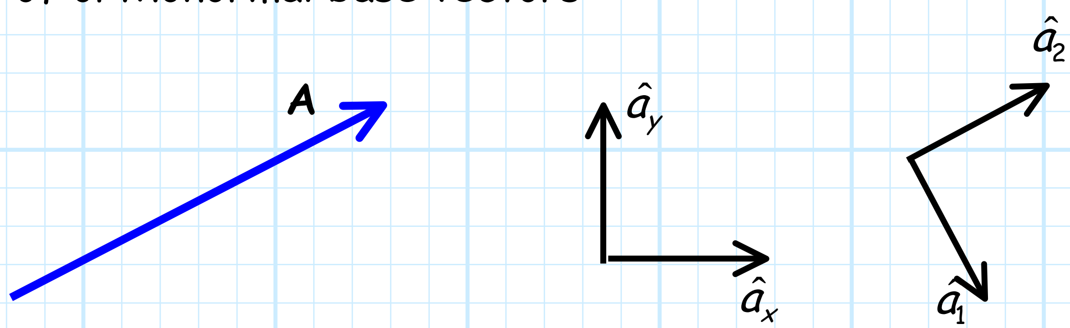
Accordingly, we find that the scalar component of vector A are determined by "doting" vector **A** with each of the three base vectors $\hat{a}_x, \hat{a}_y, \hat{a}_z$:

$$A_x = \mathbf{A} \cdot \hat{a}_x$$
$$A_y = \mathbf{A} \cdot \hat{a}_y$$
$$A_z = \mathbf{A} \cdot \hat{a}_z$$

Said another way, we **project** vector **A** onto the directions $\hat{a}_x, \hat{a}_y, \hat{a}_z$. Either way, the result is the same as determined earlier: **every** vector **A** can be expressed as a **sum** of **three** orthogonal **components**:

$$\begin{aligned}\mathbf{A} &= (\mathbf{A} \cdot \hat{a}_x)\hat{a}_x + (\mathbf{A} \cdot \hat{a}_y)\hat{a}_y + (\mathbf{A} \cdot \hat{a}_z)\hat{a}_z \\ &= A_x \hat{a}_x + A_y \hat{a}_y + A_z \hat{a}_z\end{aligned}$$

For example, consider a vector **A**, along with **two different** sets of orthonormal base vectors:

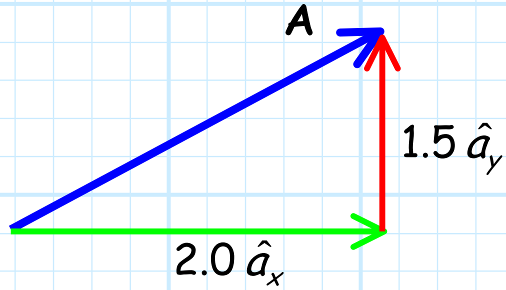


The **scalar components** of vector **A**, in the direction of each base vector are:

$$A_x = \mathbf{A}\cdot\hat{a}_x = 2.0 \qquad A_1 = \mathbf{A}\cdot\hat{a}_1 = 0.0$$
$$A_y = \mathbf{A}\cdot\hat{a}_y = 1.5 \qquad A_2 = \mathbf{A}\cdot\hat{a}_2 = 2.5$$
$$A_z = \mathbf{A}\cdot\hat{a}_z = 0.0 \qquad A_3 = \mathbf{A}\cdot\hat{a}_3 = 0.0$$

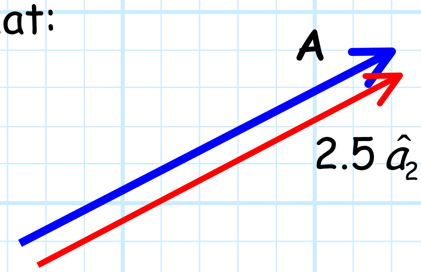
Using the **first** set of base vectors, we can write the vector **A** as:

$$\begin{aligned}\mathbf{A} &= A_x\,\hat{a}_x + A_y\,\hat{a}_y + A_z\,\hat{a}_z \\ &= 2.0\,\hat{a}_x + 1.5\,\hat{a}_y\end{aligned}$$



Or, using the **second** set, we find that:

$$\begin{aligned}\mathbf{A} &= A_1\,\hat{a}_1 + A_2\,\hat{a}_2 + A_3\,\hat{a}_3 \\ &= 2.5\,\hat{a}_2\end{aligned}$$



It is **very** important to realize that:

$$\mathbf{A} = 2.0\,\hat{a}_x + 1.5\,\hat{a}_y = 2.5\,\hat{a}_2$$

In other words, both expressions represent **exactly** the same vector! The difference in the representations is a result of using **different base vectors**, not because vector **A** is somehow "different" for each representation.